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LETTER TO THE EDITOR

A geometric approach to singularity confinement and algebraic entropy

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Abstract

A geometric approach to the equation found by Hietarinta and Viallet, which satisfies the singularity confinement criterion but exhibits chaotic behaviour, is presented. It is shown that this equation can be lifted to an automorphism of a certain rational surface and can therefore be considered to be the action of an extended Weyl group of indefinite type. A method to calculate its algebraic entropy by using the theory of intersection numbers is presented.

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1. Introduction

The singularity confinement method has been proposed by Grammaticos *et al* [1] as a criterion for the integrability of (finite- or infinite-dimensional) discrete dynamical systems. The singularity confinement method demands that even if singularities appeared due to particular initial values, such singularities would have to disappear after a finite number of iteration steps and that the information on the initial values can be recovered (hence the dynamical system has to be invertible).

However 'counter-examples' were found by Hietarinta and Viallet [2]. These mappings satisfy the singularity confinement criterion, but the orbits of their solutions exhibit chaotic behaviour. The authors of [2] introduced the notion of algebraic entropy in order to test the degree of complexity of successive iterations. The algebraic entropy is defined as $s = \lim_{n\to\infty} \log(d_n)/n$, where d_n is the degree of the nth iterate. This notion is linked with Arnold's complexity, since the degree of a mapping gives the intersection number of the image of a line and a hyperplane. While the degree grows exponentially for a generic mapping, it was shown that it grows only in the polynomial order for a large class of integrable mappings [2–5].

Many discrete Painlevé equations were found by Ramani *et al* [6, 7] and have been extensively studied. Recently it was shown by Sakai [8] that all of these (from the point of view of symmetries) are obtained by studying rational surfaces in connection with the extended affine Weyl groups. Surfaces obtained by successive blow-ups [9] of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ have been studied by several authors in the theory of birational mappings with invariants of finite (=m in the case of $\mathbb{P}^2$ and =$m-1$ in the case of $\mathbb{P}^1 \times \mathbb{P}^1$, $1 \leqslant m \leqslant 8$) point sets in a rational surface connected to the Weyl groups [10–12]. Looijenga [13] and Sakai studied the case of $m = 9$, in which case the birational mappings are connected with the extended affine Weyl groups and are obtained as Cremona transformations. Discrete Painlevé equations are recovered as particular cases.

Our aim in this Letter is to characterize one of the mappings found by Hietarinta and Viallet from the point of view of the theory of rational surfaces. As its space of initial values, we obtain a rational surface associated with some root system of indefinite type. Conversely we recover the mapping from the surface and consequently obtain an extension of mapping to its non-autonomous version. By considering the intersection numbers of divisors, we also present a method to calculate the algebraic entropy of a mapping. It is shown that the degree of the mapping is given by the nth power of a matrix which is given by the action of the mapping on the Picard group.

2. Construction of the space of initial values by blow-ups

We consider the dynamical system written by the birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$

$$(x_n, y_n) \mapsto (x_{n+1}, y_{n+1}) = (y_n, -x_n + y_n + a/y_n^2) \tag{1}$$

where $a \in \mathbb{C}$ is a nonzero constant. This equation was found by Hietarinta and Viallet [2] and we call it the HV equation. To test the singularity confinement, let us assume $x_0 \neq 0$ and $y_0 = \epsilon$ where $|\epsilon| \ll 1$. With these initial values singularities appear at $n = 1$ as $\epsilon \to 0$ and disappear at $n = 4$. In this case the information on the initial values is hidden in the coefficients of higher degree ϵ. However, taking suitable rational functions of x_n and y_n we can find the information of the initial values as finite values. The fact that the leading orders of $(x_1^2 y_1 - a)y_1$, $(x_2^3(y_2/x_2 - 1)^2 - a)x_2$ and $(x_3 y_3^2 - a)x_3$ become $-ax_0$, $-ax_0$ and $-ax_0$ actually suggests that the HV equation can be lifted to an automorphism of a suitable rational surface, although of course these rational functions are not uniquely determined.

Let the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ be (x, y), $(x, 1/y)$, $(1/x, y)$ and $(1/x, 1/y)$ and let $x = \infty$ denote $1/x = 0$. We consider the inverse mapping of the HV equation

$$\varphi^{-1} : (x, y) \mapsto (\overline{x}, \overline{y}) = (-y + x + a/x^2, x) \tag{2}$$

where $(\overline{x}, \overline{y})$ means the image of (x, y) by the mapping. This mapping has two indeterminate points: $(x, y) = (0, \infty), (\infty, \infty)$. We denote blowing up at $(x, y) = (x_0, y_0) \in \mathbb{C}^2$ by

$$(x, y) \leftarrow (x - x_0, (y - y_0)/(x - x_0)) \cup ((x - x_0)/(y - y_0), y - y_0). \tag{3}$$

By blowing up at $(x, y) = (x_0, y_0)$, $(x - x_0)/(y - y_0)$ takes meaning at this point.

First we blow up at $(x, y) = (0, \infty)$, $(x, 1/y) \leftarrow (x, 1/xy) \cup (xy, 1/y)$, and denote the obtained surface by X_0. Then φ^{-1} is lifted to a rational mapping from X_0 to $\mathbb{P}^1 \times \mathbb{P}^1$. For example, in the new coordinates φ^{-1} is expressed as

$$(u_1, v_1) := (x, 1/xy) \mapsto (\overline{x}, \overline{y}) = ((-u_1 + u_1^3 v_1 + a v_1)/(u_1^2 v_1), u_1)$$
$$(u_2, v_2) := (xy, 1/y) \mapsto (\overline{x}, \overline{y}) = ((-u_2^2 v_2 + u_2^3 v_2^3 + a)/(u_2^2 v_2^2), u_2 v_2)$$

where $u_1 = 0$ and $v_2 = 0$. This maps the exceptional curve at $(x, y) = (0, \infty)$ almost to $(\overline{x}, \overline{y}) = (\infty, 0)$ but has an indeterminate point on the exceptional curve: $(u_1, v_1) = (0, 0)$. Hence we have to blow up again at this point. In general it is known that, if there is a rational mapping $X \to Y$ where X and Y are smooth projective algebraic varieties, the procedure of blowing up can be completed in a finite number of steps, after which one obtains a smooth projective algebraic variety X_1 such that the rational mapping is lifted to a regular mapping from X_1 to Y (theorem of the elimination of indeterminacy [9]).

Here we obtain the following sequence of blow-ups (for simplicity we take only one coordinate of (3)):

$$(x, y) \xleftarrow[E_5]{\text{at } (0,\infty)} \left(x, \frac{1}{xy}\right) \xleftarrow[E_6]{(0,0)} \left(x^2 y, \frac{1}{xy}\right)$$
$$\xleftarrow[E_7]{(a,0)} \left(xy(x^2y - a), \frac{1}{xy}\right) \xleftarrow[E_8]{(0,0)} \left(x^2y^2(x^2y - a), \frac{1}{xy}\right)$$
$$(x, y) \xleftarrow[E_9]{(\infty,\infty)} \left(\frac{1}{x}, \frac{x}{y}\right) \xleftarrow[E_{10}]{(0,1)} \left(\frac{1}{x}, x\left(\frac{x}{y} - 1\right)\right)$$

where the E_i mean the total transforms generated by the blow-ups. Of course the sequence above is not unique, since there is freedom to choose the coordinates.

We have obtained a mapping from X_1 to $\mathbb{P}^1 \times \mathbb{P}^1$ which is lifted from φ^{-1}, but our aim is to construct a rational surface X such that φ^{-1} is lifted to an automorphism of X. If this can be achieved, X is considered to be the space of initial values in the sense of Okamoto [14], where a sequence of rational surfaces X_i is (or X_i themselves are) called the space of initial values for a sequence of rational mappings φ_i if each φ_i is lifted to an isomorphism from X_i to X_{i+1} for all i.

First we construct the rational surface X_2 such that φ^{-1} is lifted to a regular mapping from X_2 to X_1. For this purpose it is sufficient to eliminate the indeterminacy of mapping from X_1 to X_1. Consequently we obtain X_2 defined by the following sequence of blow-ups:

$$\left(\frac{1}{x}, x\left(\frac{x}{y} - 1\right)\right) \xleftarrow[E_{11}]{(0,0)} \left(\frac{1}{x^2(x/y - 1)}, x\left(\frac{x}{y} - 1\right)\right)$$
$$\xleftarrow[E_{12}]{(0,0)} \left(\frac{1}{x^2(x/y - 1)}, x^3\left(\frac{x}{y} - 1\right)^2\right)$$
$$\xleftarrow[E_{13}]{(0,a)} \left(\frac{1}{x^2(x/y - 1)}, x^2\left(\frac{x}{y} - 1\right)\left(x^3\left(\frac{x}{y} - 1\right)^2 - a\right)\right)$$
$$\xleftarrow[E_{14}]{(0,0)} \left(\frac{1}{x^2(x/y - 1)}, x^4\left(\frac{x}{y} - 1\right)^2\left(x^3\left(\frac{x}{y} - 1\right)^2 - a\right)\right).$$

Next eliminating the indeterminacy of mapping from X_2 to X_2, we obtain X_3 defined by the following sequence of blow-ups:

$$(x, y) \xleftarrow[E_1]{(\infty,0)} \left(\frac{1}{xy}, y\right) \xleftarrow[E_2]{(0,0)} \left(\frac{1}{xy}, xy^2\right)$$
$$\xleftarrow[E_3]{(0,a)} \left(\frac{1}{xy}, xy(xy^2 - a)\right) \xleftarrow[E_4]{(0,0)} \left(\frac{1}{xy}, x^2y^2(xy^2 - a)\right).$$

Here, it can be shown that the mapping from X_3 to X_3 which is lifted from φ^{-1} does not have any indeterminate points and has an inverse (the mapping lifted from φ). Hence we obtain the following theorem.

Theorem 2.1. *The HV equation (1) can be lifted to an automorphism of* $X\,(=X_3)$.

3. Action on the Picard group

We denote the total transform of x = constant (or y = constant) on X by H_0 (or H_1 respectively) and the total transforms of the points subjected to blow-up by $E_1, E_2, \ldots, E_{14}$. It is known [9] that the Picard group of X, $\mathrm{Pic}(X)$, is

$$\mathrm{Pic}(X) = \mathbb{Z}H_0 + \mathbb{Z}H_1 + \mathbb{Z}E_1 + \cdots + \mathbb{Z}E_{14}$$

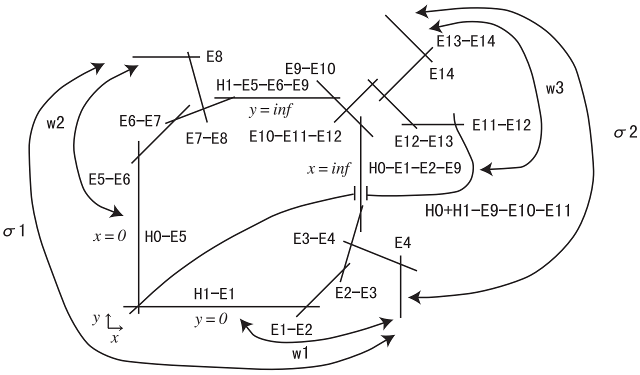


Figure 1.

and the canonical divisor of X, K_X, is

$$K_X = -2H_0 - 2H_1 + E_1 + \cdots + E_{14}.$$

It is also known that the intersection numbers of H_i or E_k and H_j or E_l are

$$H_i \cdot H_j = 1 - \delta_{i,j} \qquad E_k \cdot E_l = -\delta_{k,l} \qquad H_i \cdot E_k = 0 \tag{4}$$

where $\delta_{i,j}$ is 1 if $i = j$ and 0 if $i \neq j$.

We denote the proper transforms, i.e. prime divisors, on X by

H_0, H_1 (0 curve (the self-intersection number is 0)) :

$C_0 := E_4 \quad C_1 = E_8 \quad C_2 := E_{14} \quad C_3 := H_0 - E_5 \quad C_4 := H_1 - E_1 \quad (-1 \text{ curve})$

$D_0 := E_1 - E_2 \quad D_1 := E_2 - E_3 \quad D_2 := E_3 - E_4 \quad D_3 := E_5 - E_6 \quad D_4 := E_6 - E_7$

$D_5 := E_7 - E_8 \quad D_6 := E_9 - E_{10} \quad D_7 := E_{11} - E_{12} \quad D_8 := E_{12} - E_{13}$

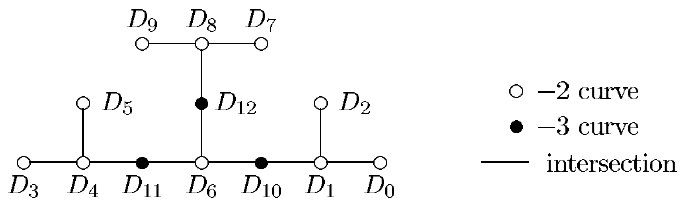
$D_9 := E_{13} - E_{14} \quad (-2 \text{ curve})$

$D_{10} := H_0 - E_1 - E_2 - E_9 \quad D_{11} := H_1 - E_5 - E_6 - E_9$

$D_{12} := E_{10} - E_{11} - E_{12} \quad (-3 \text{ curve})$

as in figure 1. The intersection numbers of any pairs of divisors are given by linear combinations of these divisors.

The anti-canonical divisor $-K_X$ can be reduced to the distinct irreducible curves $D_0, D_1, \ldots, D_{12}$ and the connection of D_i is expressed by the following Dynkin diagram:



The HV equation (1) acts on curves as

$$\begin{aligned}(D_0, D_1, D_2, D_3, D_4, D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11}, D_{12}, C_0, C_1, C_2)\\ \to (D_5, D_4, D_3, D_7, D_8, D_9, D_6, D_0, D_1, D_2, D_{11}, D_{12}, D_{10}, C_3, C_2, C_0)\end{aligned} \tag{5}$$

and $C_4 \mapsto C_1$. Hence the HV equation acts on Pic(X) as

$$\begin{pmatrix} H_0 \\ H_1, E_1, E_2 \\ E_3, E_4, E_5, E_6 \\ E_7, E_8, E_9, E_{10} \\ E_{11}, E_{12}, E_{13}, E_{14} \end{pmatrix} \to \begin{pmatrix} 3H_0 + H_1 - E_5 - E_6 - E_7 - E_8 - E_9 - E_{10} \\ H_0, H_0 - E_8, H_0 - E_7 \\ H_0 - E_6, H_0 - E_5, E_{11}, E_{12} \\ E_{13}, E_{14}, H_0 - E_{10}, H_0 - E_9 \\ E_1, E_2, E_3, E_4 \end{pmatrix} \tag{6}$$

(this table means $\overline{H_0} = 3H_0 + H_1 - E_5 - E_6 - E_7 - E_8 - E_9 - E_{10}$, $\overline{H_1} = H_0$, $\overline{E_1} = H_0 - E_8$ and so on) and their linear combinations. As we will see in section 6, the action (6) provides a method to calculate the algebraic entropy of the HV equation.

4. An extended Weyl group acting on the Picard group

We shall decompose the action of the HV equation on Pic(X) as a product of actions of order two elements of what turns out to be an extended Weyl group. Let us define the actions $\sigma_1, \sigma_2, w_1, w_2, w_3$ on Pic(X) as follows (see figure 1) (for simplicity we have not written the invariant elements under each action):

$$\begin{aligned}
\sigma_1 :&\quad \begin{pmatrix} H_0, H_1, E_1, E_2, E_3 \\ E_4, E_5, E_6, E_7, E_8 \end{pmatrix} \to \begin{pmatrix} H_1, H_0, E_5, E_6, E_7 \\ E_8, E_1, E_2, E_3, E_4 \end{pmatrix} \\
\sigma_2 :&\quad \begin{pmatrix} H_1, E_1, E_2 \\ E_3, E_4, E_9, E_{10} \\ E_{11}, E_{12}\, E_{13}, E_{14} \end{pmatrix} \to \begin{pmatrix} H_0 + H_1 - E_9 - E_{10}, E_{11}, E_{12} \\ E_{13}, E_{14}, H_0 - E_{10}, H_0 - E_9 \\ E_1, E_2, E_3, E_4 \end{pmatrix} \\
w_1 :&\quad \begin{pmatrix} H_0, \\ E_1, E_2, E_3, E_4 \end{pmatrix} \to \begin{pmatrix} H_0 + 2H_1 - E_1 - E_2 - E_3 - E_4 \\ H_1 - E_4, H_1 - E_3, H_1 - E_2, H_1 - E_1 \end{pmatrix} \\
w_2 :&\quad \begin{pmatrix} H_1, \\ E_5, E_6, E_7, E_8 \end{pmatrix} \to \begin{pmatrix} 2H_0 + H_1 - E_5 - E_6 - E_7 - E_8 \\ H_0 - E_8, H_0 - E_7, H_0 - E_6, H_0 - E_5 \end{pmatrix} \\
w_3 :&\quad \begin{pmatrix} H_0, H_1, E_9, E_{10} \\ E_{11}, E_{12}, E_{13}, E_{14} \end{pmatrix} \to \begin{pmatrix} H_0 + \alpha_3, H_1 + \alpha_3, E_9 + \alpha_3, E_{10} + \alpha_3 \\ E_{11} + \alpha_3^1, E_{12} + \alpha_3^2, E_{13} + \alpha_3^2, E_{14} + \alpha_3^1 \end{pmatrix}
\end{aligned} \tag{7}$$

where $\alpha_3^1 = H_0 + H_1 - E_9 - E_{10} - E_{11} - E_{14}$, $\alpha_3^2 = H_0 + H_1 - E_9 - E_{10} - E_{12} - E_{13}$ and $\alpha_3 = \alpha_3^1 + \alpha_3^2$.

Then (6) becomes $w_2 \circ \sigma_2 \circ \sigma_1$ and the following relations hold:

$$\begin{aligned}
&w_i^2 = \sigma_j^2 = 1 && (\sigma_1\sigma_2)^3 = 1 && \\
&\sigma_1 w_1 = w_2 \sigma_1 && \sigma_1 w_2 = w_1 \sigma_1 && \sigma_1 w_3 = w_3 \sigma_1 \\
&\sigma_2 w_1 = w_3 \sigma_2 && \sigma_2 w_2 = w_2 \sigma_2 && \sigma_2 w_3 = w_1 \sigma_2.
\end{aligned} \tag{8}$$

The basis of the root system. Let us define $\alpha_1, \alpha_2\, \alpha_3 \in$ Pic(X) as

$$\begin{aligned}
\alpha_1 &= 2H_1 - E_1 - E_2 - E_3 - E_4 \\
\alpha_2 &= 2H_0 - E_5 - E_6 - E_7 - E_8 \\
\alpha_3 &= 2H_0 + 2H_1 - 2E_9 - 2E_{10} - E_{11} - E_{12} - E_{13} - E_{14}.
\end{aligned} \tag{9}$$

It is satisfied that $\alpha_i \cdot D_j = 0$ for all j. The actions of σ_j and w_k are

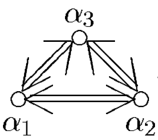
$\sigma_j(\alpha_i)$ and $w_k(\alpha_i)$

	σ_1	σ_2	w_1	w_2	w_3
$\alpha_1 \mapsto$	α_2	α_3	$-\alpha_1$	$\alpha_1 + 2\alpha_2$	$\alpha_1 + 2\alpha_3$
$\alpha_2 \mapsto$	α_1	α_2	$\alpha_2 + 2\alpha_1$	$-\alpha_2$	$\alpha_2 + 2\alpha_3$
$\alpha_3 \mapsto$	α_3	α_1	$\alpha_3 + 2\alpha_1$	$\alpha_3 + 2\alpha_2$	$-\alpha_3$

(10)

The action of w_i can be written in the form $w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$ where $c_{ij} = 2(\alpha_i \cdot \alpha_j)/(\alpha_i \cdot \alpha_i)$. Its Cartan matrix and Dynkin diagram are of the indefinite type $H_{71}^{(3)}$ [15]:

$$\begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \quad \text{and} \tag{11}$$



Hence it is seen [16] that the group of actions on Pic(X) generated by w_i and σ_i coincides with the extended (including the full automorphism group of the Dynkin diagram) Weyl group of an indefinite type generated by

$$\langle w_1, w_2, w_3, \sigma_1, \sigma_2 \rangle \tag{12}$$

and the fundamental relations (8). From this fact we have the following theorem.

Theorem 4.1. *The HV equation as the action of $w_2\sigma_2\sigma_1$ on* Pic(X) *does not commute with any element of the group generated by w_i and σ_i except $(w_2\sigma_2\sigma_1)^m$.*

5. The inverse problem

A birational mapping on a rational surface is called a Cremona transformation. One method for obtaining a Cremona transformation (which exchanges a certain pair of exceptional curves) is to interchange the blow-down structures. Following this method, we can construct the Cremona transformations which yield the extended Weyl group (12) and thereby recover the HV equation from its action on Pic(X).

These Cremona transformations are lifted to automorphisms of Pic(X) but do not have to be lifted to automorphisms of X itself, i.e. the blow-up points can be changed without however changing the intersection numbers (we consider isomorphisms from X to X', where X and X' may have different blow-up points). In order to do this, one has to consider not only autonomous but also non-autonomous mappings. By $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7$, we denote the point where each $E_{10}, E_3, E_4, E_7, E_8, E_{11}, E_{13}, E_{14}$ is generated by the blow-up (or its value of the coordinate).

Consequently, it can been seen that w_2 is written as $w_2 = t \circ w_2'$, where $w_2' : (x, y) \mapsto (x, y - a_3/x^2 - a_4/(a_3x))$, and t is a certain automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. By taking suitable t, we can assume that the points of first, fifth and ninth blow-up are fixed, $\overline{a_0} = a_0 = 1$ and $\overline{a_5} = a_5$. For the remaining points there are no such *a priori* requirements and there evolution under the present isomorphism should be calculated in detail. For example, under the above assumptions, w_2 can be seen to reduce to

$$\begin{aligned} w_2 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) &\mapsto (\overline{x}, \overline{y}; \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}, \overline{a_6}, \overline{a_7}) \\ &= \left(x, y - \frac{a_3}{x^2} - \frac{a_4}{a_3 x}; a_1, a_2 - \frac{2a_1a_4}{a_3}, -a_3, a_4, a_5, a_6, a_7 + \frac{2a_4a_6}{a_3}\right). \end{aligned} \tag{13}$$

Here, in the calculation of the next iteration step we have to use $\overline{a_3} = -a_3$ instead of a_3.

Similarly σ_1 and σ_2 reduce to

$$\sigma_1 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto (-y, -x; -a_3, -a_4, -a_1, -a_2, a_5, -a_6, a_7 - 4a_5^2a_6)$$

and

$$\begin{aligned} \sigma_2 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto (&x, x - y - a_5; \\ &a_6, a_7 - 2a_5^2a_6, -a_3, a_4, a_5, a_1, a_2 + 2a_1a_5^2). \end{aligned}$$

Non-autonomous HV equation. The composition $w_2\sigma_2\sigma_1$ reduces to

$$w_2 \circ \sigma_2 \circ \sigma_1 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto \bigg(- y, x - y - a_5 - \frac{a_1}{y^2} - \frac{a_2}{a_1 y};$$

$$-a_6, a_7 - 2a_5^2 a_6 - \frac{2a_2 a_6}{a_1}, -a_1, -a_2, a_5, -a_3, -a_4 - 2a_3 a_5^2 + \frac{2a_2 a_3}{a_1}\bigg). \quad (14)$$

Of course this mapping satisfies the singularity confinement criterion by construction and in the case of $a_2 = a_4 = a_5 = a_7 = 0$ and $a_1 = a_3 = a_6 = a$ it coincides with the HV equation (1) except their signs. The difference between them comes from the assumption $\overline{a_5} = a_5$. Assuming $\overline{a_5} = -a_5$ by w_2, σ_2 and σ_1, we have another form of (14)

$$w_2 \circ \sigma_2 \circ \sigma_1 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto \bigg(y, -x + y + a_5 + \frac{a_1}{y^2} + \frac{a_2}{a_1 y};$$

$$a_6, -a_7 + 2a_5^2 a_6 + \frac{2a_2 a_6}{a_1}, a_1, a_2, -a_5, a_3, a_4 + 2a_3 a_5^2 - \frac{2a_2 a_3}{a_1}\bigg). \quad (15)$$

Actually in the case of $a_2 = a_4 = a_5 = a_7 = 0$ and $a_1 = a_3 = a_6 = a$ it coincides with the HV equation (1).

6. Algebraic entropy

In this section we consider the algebraic entropy which has been introduced by Hietarinta and Viallet to describe the complexity of rational mappings [2]. The degree of a rational function $P(x, y) = f(x, y)/g(x, y)$, where $f(x, y)$ and $g(x, y)$ are polynomials and $P(x, y)$ is irreducible, is defined by

$$\deg(P) = \max\{\deg f(x, y), \deg g(x, y)\}$$

where $\deg(x^m y^n) = m + n$. The degree of the mapping $\varphi : (x, y) \mapsto (P(x, y), Q(x, y))$, where $P(x, y)$ and $Q(x, y)$ are rational functions, is defined by

$$\deg(\varphi) = \max\{\deg P(x, y), \deg Q(x, y)\}.$$

The algebraic entropy of the map $\varphi : (x, y) \mapsto (P(x, y), Q(x, y))$ is defined by

$$\lim_{n\to\infty} \frac{1}{n} \log \deg(\varphi^n)$$

if the limit exists. The definition of algebraic entropy of φ coincides with the definition for the case where φ is a rational mapping from $\mathbb{P}^2 \to \mathbb{P}^2$. It is known [2] that the algebraic entropy of the HV equation becomes $\log(3 + \sqrt{5})/2$. We recover the algebraic entropy of the HV equation by using the theory of intersection numbers.

Let us define the curve D in X as $y/x = c$, where $c \in \mathbb{C}$ is a nonzero constant. We denote the HV equation by φ and (x_n, y_n) by $(P_n(x_0, y_0), Q_n(x_0, y_0))$. By the fundamental theorem of algebra, $\deg_t(P_n(t, ct))(= \deg(P_n(x, y))$ for $c \neq 1)$ coincides with the intersection number of the curve $x = P_n(t, ct)$ and the curve $x = d$ in $\mathbb{P}^1 \times \mathbb{P}^1$, where $\deg_t(P(t))$ is the degree of the rational function of one variable $P(t)$ and $d \in \mathbb{C}$ is a nonzero constant. It also coincides with the coefficient of H_1 as an element of $\mathrm{Pic}(X)$. (Analogously the intersection number of the curves related to Q coincides with the coefficient of H_0.) The curve D is expressed as $H_0 + H_1 - E_9$ in $\mathrm{Pic}(X)$ if $c \neq 1$. Hence writing the coefficients of H_0 and H_1 of $\varphi^n(H_0 + H_1 - E_9)$ as h_n^0, h_n^1, we obtain the formulae

$$\deg(P_n) = h_n^1 \qquad \deg(Q_n) = h_n^0.$$

The action of φ on Pic(X) is given by (6). Hence the algebraic entropy of the HV equation becomes $\lim_{n\to\infty}\frac{1}{n}\log\max\{h_n^0, h_n^1\}$. In this case we have

$$\log\max\{|\text{ eigenvalues of }\varphi|\} = \log\frac{3+\sqrt{5}}{2}.$$

The proof of the theorems contained in this Letter as well as some other mappings, which satisfy the singularity confinement criterion but which have positive algebraic entropy, will be discussed in a forthcoming paper.

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